ISRAEL JOURNAL OF MATHEMATICS, Vol. 52, Nos. 1-2, 1985

ZARISKI DENSITY IN LIE GROUPS

BY
RICHARD D. MOSAK[a] AND MARTIN MOSKOWITZ[b]
[a]*Department of Mathematics and Computer Science, Lehman College, CUNY, Bronx, NY 10468, USA; and* [b]*Department of Mathematics, Graduate Center, CUNY, New York, NY 10036, USA*

ABSTRACT

In [7] Furstenberg gave a proof of Borel's density theorem [1], which depended not on complete reducibility but rather on properties of the action of a minimally almost periodic group on projective space. In [9] and [10] the basic idea of this proof was extended in various ways to deal with other particular classes of Lie groups G and closed subgroups H of cofinite volume. In [5] Dani gives a more general form of the density theorem in which H need only be non-wandering. In the present paper we define the condition of k-minimal quasiboundedness, and prove that this condition is necessary and sufficient for the density theorem to hold ((2.4) and (2.6)). Here we replace the arguments of [9] and [10] simply by proofs that the groups considered there satisfy this condition (2.10). We extend the results of [9] and [10] by considering groups which are analytic rather than algebraic, and in the solvable case we completely characterize the k-minimally quasibounded groups (2.9). In the last section we give two applications of the density theorem.

§1. Throughout this paper V will denote a finite-dimensional vector space over k, where k is either $\mathbf{R}$ or $\mathbf{C}$, and $\|\ \|$ will denote a Banach algebra norm on $\mathrm{End}_k(V)$ (i.e., a submultiplicative norm with $\|I\| = 1$). If $x \in \mathrm{GL}(V)$, x is said to be *bounded* if the cyclic subgroup $\langle x \rangle$ generated by x is a bounded subgroup of $\mathrm{GL}(V)$ (equivalently, if the closure of $\langle x \rangle$ in $\mathrm{End}_k(V)$ is compact). We shall need a more general notion, and for this purpose we define $\omega(x) = \|x\|^n / |\det x|$, where $n = \dim V$. Then ω is a continuous function on $\mathrm{GL}(V)$, satisfying $\omega(xy) \leqq \omega(x)\omega(y)$ and $\omega(x) \geqq 1$ for every $x, y \in \mathrm{GL}(V)$ (the latter inequality follows from the fact that the spectral radius $\sigma(x) = \max\{|\lambda| : \lambda$ is an eigenvalue of $x\}$ satisfies $\sigma(x) \leqq \|x\|$). Now we shall say that x is *quasibounded* if $\omega|_{\langle x \rangle}$ is bounded; more generally, if G is a subgroup of $\mathrm{GL}(V)$, and $\omega|_G$ is bounded, we shall say that G is *quasibounded.*

Received March 13, 1984 and in revised form February 18, 1985

(1.1) REMARKS.

(1) The notion of quasiboundedness is independent of the particular choice of norm on $\mathrm{End}_k(V)$, since all such norms are equivalent.

(2) The notion of quasiboundedness is preserved under extension or restriction of the field of scalars, in the following sense. If $k = \mathbf{R}$, and $V_{\mathbf{C}}$ denotes the complexification of V, then we may choose a Banach algebra norm $\| \ \|_{\mathbf{C}}$ on $\mathrm{End}_{\mathbf{C}}(V_{\mathbf{C}})$ so that $\|x_{\mathbf{C}}\|_{\mathbf{C}} = \|x\|$ for all $x \in \mathrm{GL}(V)$ (here $x_{\mathbf{C}}$ denotes the extension of x to $V_{\mathbf{C}}$). Thus $\omega_{V_{\mathbf{C}}}(x_{\mathbf{C}}) = \omega_V(x)$. On the other hand, if $k = \mathbf{C}$, and $V^{\mathbf{R}}$ denotes the real $2n$-dimensional space obtained by restricting scalars, then $\|x^{\mathbf{R}}\| = \|x\|$ while $\det x^{\mathbf{R}} = |\det x|^2$ (since the eigenvalues of $x^{\mathbf{R}}$ consist of those of x together with their complex conjugates.) Thus $\omega_V \mathbf{R}(x^{\mathbf{R}}) = \omega_V(x)^2$ for all $x \in \mathrm{GL}(V)$, so $G \subset \mathrm{GL}(V)$ is quasibounded iff $G^{\mathbf{R}} \subset \mathrm{GL}(V^{\mathbf{R}})$ is quasibounded.

(3) If $H \subset G$ are subgroups of $\mathrm{GL}(V)$, and H has finite index in G, then G is quasibounded iff H is quasibounded: this follows from submultiplicativity of ω and the fact that $G = FH$ for some finite set F. Obviously it would be sufficient for F to be compact; thus the same result holds if $H \subset G$ are Lie subgroups of $\mathrm{GL}(V)$ with H closed in G and G/H compact (in the Lie topology).

(4) $\omega(\lambda x) = \omega(x)$ for all $x \in \mathrm{GL}(V)$, $\lambda \in k^\times$.

(1.2) LEMMA. *Let $x \in \mathrm{GL}(V)$. Then the following are equivalent*:

(i) *x is quasibounded.*

(ii) *x is semisimple, and all the eigenvalues of x have the same modulus.*

(iii) *$x = \rho y$, for some bounded $y \in \mathrm{GL}(V)$ and some $\rho > 0$; in this case $\rho = |\det y|^{1/n}$.*

(iv) *$x^m = \rho y$, for some $m \in \mathbf{N}$, some bounded $y \in \mathrm{GL}(V)$, and some $\rho > 0$. If x lies on a one-parameter group $(e^{tX})_{t \in \mathbf{R}}$, then these conditions are equivalent to*

(v) *$X = \alpha I + Y$, where $\alpha \in \mathbf{R}$ and Y is semisimple with purely imaginary eigenvalues.*

PROOF. If $k = \mathbf{C}$, and $x^{\mathbf{R}}$ is the operator on $V^{\mathbf{R}}$ obtained by restricting scalars, then it is easy to see that x is semisimple iff $x^{\mathbf{R}}$ is semisimple, and as we have noted, the eigenvalues of $x^{\mathbf{R}}$ consist of those of x together with their conjugates. Thus we may assume without loss of generality that $k = \mathbf{R}$.

(i) $\Rightarrow$ (ii): Since ω is submultiplicative and bounded on $\langle x \rangle$ we have

$$1 = \lim \omega(x^j)^{1/j} = \lim \frac{\|x^j\|^{n/j}}{|\det x^j|^{1/j}} = \frac{\sigma(x)^n}{|\det x|}.$$

Therefore $|\det x| = \sigma(x)^n$ so that $|\lambda| = \sigma(x)$ for all eigenvalues λ of x. It follows that if we write x as a commuting product $x = ehu$ with e elliptic, h hyperbolic, and u unipotent, then $h = \sigma(x)I = \rho I$ is scalar with $\rho > 0$. Furthermore, since ω is submultiplicative, ω is bounded on the set of elements $u^j = \rho^{-j}e^{-j}x^j$. Since u is unipotent, $\det u^j = 1$, so $\langle u \rangle$ must be bounded, which is impossible unless $u = 1$. Therefore, $x = \rho e$ is semisimple.

(ii) ⇒ (iii): The unipotent part of x is trivial, so the argument above shows that $x = \rho e$ with $h = \rho I = \sigma(x)I$.

(iii) ⇒ (iv): obvious.

(iv) ⇒ (i): The hypothesis implies that ω is bounded on the subgroup $\langle x^m \rangle$, which has finite index in $\langle x \rangle$. Therefore the conclusion follows from Remark (1.1)(3).

Now suppose that $x = e^X$. Then (v) certainly implies (i). On the other hand, since the subgroup $(e^{nX})_{n\in\mathbf{Z}}$ of $(e^{tX})_{t\in\mathbf{R}}$ is quasibounded, Remark (1.1)(3) again implies that $(e^{tX})_{t\in\mathbf{R}}$ is quasibounded. Thus (i) holds for each $t \in \mathbf{R}$, and (v) follows easily. ■

REMARK. In [5] Dani has called an element of $\mathrm{GL}(V)$ *projectively bounded* if it satisfies condition (iv) above, and has remarked in [6] that (iv) and (iii) are equivalent.

(1.3) LEMMA. *Let G be an abelian subgroup of* $\mathrm{GL}(V)$, *consisting entirely of quasibounded elements. Then the group $B = \{x/|\det x|^{1/n} : x \in G\}$ is bounded. Thus there is a compact abelian group $C = \bar{B} \subset \mathrm{GL}(V)$ such that every element $x \in G$ is of the form $x = \rho c$ with $\rho > 0$ and $c \in C$.*

PROOF. We may obviously assume that $k = \mathbf{C}$, since otherwise we could complexify V. Since by Lemma (1.2) all the elements of G are semisimple, they may be simultaneously diagonalized relative to some basis in V. Condition (ii) of Lemma (1.2) now shows that relative to this basis, $B \subset U(n) \cap D(n)$. ■

(1.4) COROLLARY. *Let G be a solvable subgroup of* $\mathrm{GL}(V)$, *consisting entirely of quasibounded elements. Then G is quasibounded, and contains an abelian subgroup of finite index. In particular, if G is connected then it is abelian.*

PROOF. By the Lie–Kolchin theorem [2, p. 243], a subgroup $H \subset G$ of finite index can be put into triangular form over $\mathbf{C}$. Thus every element of (H, H) is unipotent, but also semisimple by Lemma (1.2), so H is abelian. By Lemma (1.3), $\omega(H) \subset \omega(C)$ for a certain compact group C, so H is quasibounded, hence also G (Remark (1.1)(3)). ■

(1.5) PROPOSITION. *Let G be a real Lie subgroup of* $\mathrm{GL}(V)$, *and suppose G is almost connected, that is, G_0 (the connected component of* 1) *has finite index in G. Then the following are equivalent*:

(i) *G is quasibounded.*

(ii) *Every element of G is quasibounded.*

(iii) *If $G_0 = S \cdot R$ is the Levi decomposition of G_0 (with S semisimple and R the radical), then S is compact, R is central in G_0, and R is quasibounded.*

(iv) *G contains a closed, quasibounded subgroup A such that A is (isomorphic to) a vector group, and G/A is compact.*

PROOF. (i) $\Rightarrow$ (ii): Obvious.

(ii) $\Rightarrow$ (iii): Let $G_0 = S \cdot R$ be the Levi decomposition. Since S is semisimple, $\det x = 1$ for every $x \in S$, so condition (ii) of (1.2) implies that every element of S is bounded. By semisimplicity, S is closed in $\mathrm{GL}(V)$ and therefore compact. Next, by (1.4) R is abelian and quasibounded. If B and $C = \bar{B}$ are the subgroups of (1.3), then S normalizes B (since R is normal) and hence C. Since C has a discrete automorphism group and S is connected, S centralizes C and therefore R.

(iii) $\Rightarrow$ (iv): Since $R = H \cdot A$ (direct product) with H compact and A a vector group, it follows that $G_0 = SH \cdot A = K \cdot A$ with K compact; thus G/A is compact.

(iv) $\Rightarrow$ (i): Remark (1.1)(3).

(1.6) COROLLARY. *An almost connected, quasibounded real Lie subgroup G of $\mathrm{GL}(V)$ acts semisimply on V.*

PROOF. By condition (iii) above, the Lie algebra of G is reductive, so G_0 acts semisimply. Hence G also acts semisimply, by [11]. ∎

(1.7) COROLLARY. *Let $k = \mathbf{R}$, and let G be a quasibounded vector group in $\mathrm{GL}(V)$. Then there is a basis of V relative to which the matrices of elements of G have simultaneous block diagonal form, in which each block is either 1×1 or 2×2. More precisely, if (e^{tX}) is a one-parameter group in G, then there exist $\alpha, \beta_1, \ldots, \beta_m \in \mathbf{R}$ such that the matrix of e^{tX} has the form*

$$e^{\alpha t}\begin{pmatrix} R_1(t) & & \\ & \ddots & \\ & & R_m(t)\end{pmatrix}$$

where each $R_j(t)$ is either a 2×2 block of the form

$$R_j(t) = \begin{pmatrix} \cos\beta_j t & -\sin\beta_j t \\ \sin\beta_j t & \cos\beta_j t\end{pmatrix}$$

or $R_j(t)$ is the unit 1×1 block (and $\beta_j = 0$). Thus each quasibounded vector group acts as a family of generalized spirals.

PROOF. This follows easily from (1.6) and (1.2). ∎

We conclude this section with a simple lemma.

(1.8) LEMMA. *Let G be a Lie subgroup of* $\mathrm{GL}(V)$ *all of whose elements are quasibounded. Then G is unimodular.*

PROOF. Let $\mathfrak{G}$ be the real Lie algebra of G, regarded as a subalgebra of $\mathfrak{G}l(V)$. By [8, p. 366] the modular function $\Delta_G(g) = |\det \mathrm{Ad}_G(g)|$ for all $g \in G$; thus it suffices to prove that $|\det \mathrm{Ad}_G(\langle g \rangle)|$ is a bounded subgroup of $\mathbf{R}_+^\times$. But $g = \rho u$ with $\rho > 0$ and u bounded, so (since G is linear) for all $M \in \mathfrak{G}$ we have

$$\mathrm{Ad}_G(g)(M) = gMg^{-1} = uMu^{-1} = \alpha(u)(M), \qquad \text{where } \alpha(u) = \mathrm{Ad}_{\mathrm{GL}(V)}(u)|\mathfrak{G}.$$

Thus the group $\mathrm{Ad}_G(\langle g \rangle)$ is a subgroup of the compact group $\alpha(\langle u \rangle)$. ∎

§2. In order to apply the previous results to questions of Zariski density, we shall need some definitions. If V is a finite-dimensional vector space over k ($k = \mathbf{R}$ or $\mathbf{C}$) and G is a subgroup of $\mathrm{GL}(V)$, then $G^{\#}$ denotes the algebraic hull of G in $V_{\mathbf{C}}$ (where $V_{\mathbf{C}}$ denotes V if $k = \mathbf{C}$, and $V_{\mathbf{C}}$ denotes the complexification of $V = V_k$ if $k = \mathbf{R}$). $G^{\#}$ is of course defined over k. If H is a subgroup of G, then H is *Zariski dense* in G if $H^{\#} = G^{\#}$. If W is another finite-dimensional space, then a homomorphism $\pi : G \to \mathrm{GL}(W)$ will be called a *k-rational representation* if π is the restriction to G of an algebraic k-group morphism $\pi^{\#} : G^{\#} \to \mathrm{GL}(W_{\mathbf{C}})$. We shall need terms for the groups which are at the opposite end of the spectrum from the bounded and quasibounded groups discussed in §1. We shall say that G is *k-minimally almost periodic* if it has essentially no rational bounded representations, and *k-minimally quasibounded* if it has essentially no rational quasibounded representations. More precisely, we make the following definitions.

(2.1) DEFINITION. A group $G \subset \mathrm{GL}(V)$ is *k-minimally almost periodic* if whenever π is a k-rational representation of G for which $\pi(G)$ is bounded, then $\pi(G) = \{I\}$. G is *k-minimally quasibounded* if whenever π is a k-rational representation of G for which $\pi(G)$ is quasibounded, then $\pi(G) \subset kI$.

Our main results of this section (2.2, 2.6) are that the (generalized) Borel density theorem holds for k-minimally quasibounded groups, and that among the almost connected groups (for example, the groups $G_{\mathbf{R}}$ of real points of an $\mathbf{R}$-algebraic group) the density theorem holds *only* for the k-minimally quasibounded groups. We shall see later (2.9, 2.10, 2.11) some examples of these groups, and state what we know about the relationship between the two notions of minimal quasiboundedness and minimal almost periodicity (2.5). For now we need two preliminary lemmas.

(2.2) LEMMA. *Let $G \subset \mathrm{GL}(V)$. If G is k-minimally quasibounded then G is k-minimally almost periodic.*

PROOF. Let $\pi : G \to \mathrm{GL}(W)$ be a k-rational representation with $\pi(G)$ bounded, and let $\sigma : G \to \mathrm{GL}(W_{\mathbf{C}} \oplus \mathbf{C})$ be defined by $\sigma(g) = \pi(g) \oplus 1$. Then σ is a k-rational morphism with $\sigma(G)$ quasibounded (in fact, bounded), so $\sigma(G)$ consists of scalars. Therefore $\pi(G) = \{I\}$.

(2.3) LEMMA. *Let $G \subset \mathrm{GL}(V)$.*

(i) *$G^{\#}$ is connected iff all subgroups of finite index in G are Zariski dense in G.*

(ii) *If G is k-minimally almost periodic then $G^{\#}$ is connected.*

PROOF.

(i) Suppose $G^{\#}$ is connected and $H \subset G$ has index n, so $G = \bigcup_{i=1}^{n} x_i H$. Then $G^{\#} = \bigcup_{i=1}^{n} x_i H^{\#}$, so $(G^{\#} : H^{\#}) \leqq n$. Since $G^{\#}$ is connected, $H^{\#} = G^{\#}$. Conversely, suppose all subgroups of finite index in G are Zariski dense in G. If $G_0^{\#}$ denotes the identity component of $G^{\#}$, then

$$(G : G \cap G_0^{\#}) = (GG_0^{\#} : G_0^{\#}) \leqq (G^{\#} : G_0^{\#}) < \infty.$$

Then by hypothesis the hull of $G \cap G_0^{\#}$ is $G^{\#}$, so $G_0^{\#}$ is also Zariski dense in $G^{\#}$. Hence $G_0^{\#} = G^{\#}$.

(ii) Since $G^{\#}$ and $G_0^{\#}$ are defined over k, there is a k-rational morphism $\tau : G^{\#} \to \mathrm{GL}(E_{\mathbf{C}})$ such that $\ker \tau = G_0^{\#}$; thus $\tau(G^{\#})$ is finite and *a fortiori* $\tau(G)$ is finite. Therefore, $\tau(G) = \{I\}$, so $G \subset G_0^{\#}$, and $G^{\#} = G_0^{\#}$ is connected. ∎

We remark parenthetically that one can see easily from the Lemma (or otherwise) that $G^{\#}$ is connected iff G satisfies the following conditions: whenever π is a k-rational representation of G for which $\pi(G)$ is *finite*, then $\pi(G) = \{I\}$. Looked at from this perspective, the notions of k-minimal almost periodicity and k-minimal quasiboundedness turn out to be strong connectedness conditions on G.

(2.4) THEOREM. *Let G be a k-minimally quasibounded Lie subgroup of $\mathrm{GL}(V)$, and let H be a closed subgroup of G such that G/H carries a finite invariant measure.*

(i) *If $\pi : G \to \mathrm{GL}(W)$ is a k-rational representation, then each H-invariant subspace U of W is G-invariant.*

(ii) *H is Zariski dense in G.*

PROOF. Every r-dimensional subspace of W is a point of the Grassman space $\mathcal{G}^r(W)$, and the map $\mathcal{G}^r(W) \to P(\Lambda^r W)$ is G-equivariant and injective. Replacing W by $\Lambda^r W$ and π by the k-rational representation $\Lambda^r \pi$, we see that it suffices to prove the result when $U = L$ is one-dimensional. Furthermore, since G is k-minimally quasibounded, it will suffice to prove that there is a G-invariant subspace $W_1 \supset L$ such that $\pi(G)|_{W_1}$ is quasibounded: for in this case $\pi(G)$ will act by scalars on W_1, hence also on L (cf. [5, Theorem 2.1] as well as the original arguments in [7] and [9]).

If $p = \bar{L} \in P(W)$, where $x \mapsto \bar{x}$ denotes the map $W \to P(W)$, then ([9, Lemma 1.1]), there is a minimal quasilinear variety $X = \bar{W}_1 \cup \cdots \cup \bar{W}_n$ in $P(W)$ containing the orbit $\mathcal{O} = \pi(G)p$ (W_i is a subspace of W for $i = 1, \ldots, n$). Assuming, as we may, that $L \subset W_1$ and no W_i is redundant, it follows (cf. [9, Lemma 1.3]) that G permutes the spaces W_i transitively, and that $G_1 = \{g \in G : \pi(g)W_1 = W_1\}$ is a subgroup of index n in G. But since $G^{\#}$ is connected by (2.2) and (2.3), it follows from (2.3) that G_1 is Zariski dense in G, so G leaves W_1 invariant; hence $G = G_1$, and $X = W_1$ is the smallest quasilinear variety in $P(W)$ containing $\mathcal{O}$. Now for $g \in G$, let $\pi_1(g) = \pi(g)|_{W_1}$. If $\pi_1(G)$ is not quasibounded, then by [7, Lemma 2] or [9, Lemma 1.4] there is a sequence (σ_k) in $\pi_1(G)$ such that the projective transformations $\bar{\sigma}_k$ on $P(W_1)$ converge pointwise to a map $\varphi : P(W_1) \to P(W_1)$, and $Y = \varphi(P(W_1))$ is a proper quasilinear variety in $X = P(W_1)$. Now by [7, Lemma 3] or [9, Lemma 1.5], any finite G-invariant measure on X is supported in Y. But since G/H carries a finite invariant measure μ, the image of μ under the map $G/H \to \mathcal{O}$ can be regarded as a G-invariant measure on $P(W_1)$ whose support contains $\mathcal{O}$. Therefore $\mathcal{O} \subset Y \subsetneq X$, contradicting minimality of X. Thus $\pi(G)|_{W_1}$ is quasibounded, which completes the proof of (i).

(ii) There is a k-space W_C and a line $L_C \subset W_C$ defined over k, and a k-morphism $\pi^{\#} : G^{\#} \to \mathrm{GL}(W_C)$ for which $H^{\#} = \{g \in G^{\#} : \pi^{\#}(g)L_C = L_C\}$ [2, Theorem 5.1]. Then $\pi = \pi^{\#}|_G$ is a k-rational representation of G on $W = W_k$, and the line $L = L_k$ is $\pi(H)$-stable. By (i), L is $\pi(G)$-stable, so $G \subset H^{\#}$. ■

For ease of comparison of the present Borel density theorem (2.2) with earlier versions in [7], [9], and [10], we have chosen to maintain the hypothesis used there, that G/H have finite volume. The proof of (2.2.i) in conjunction with Dani's [5, Theorem 2.1] shows, however, that the theorem would still be true if H were only assumed to be a non-wandering subgroup of G.

Before stating our converse to (2.4), we need the following extension of Lemma (2.2).

(2.5) PROPOSITION. *Let G be a subgroup of* $\mathrm{GL}(V)$. *If G is k-minimally quasibounded, then G is k-minimally almost periodic. The converse holds if G is solvable, or if G is an almost connected k-Lie subgroup of* $\mathrm{GL}(V)$.

PROOF. The first part has been proved in (2.2). To prove the converse, let $\pi : G \to \mathrm{GL}(W)$ be a k-rational representation for which $M = \pi(G)$ is quasibounded. We must show that $M \subset kI$, and to do this, we claim, it is sufficient to prove that M is abelian. For in this case, the map $m \mapsto \varphi(m) = m^d/\det(m)$ (where $d = \dim W$) is a k-rational representation mapping M into a compact group (1.3). Thus $\varphi(M) = (\varphi \circ \pi)(G)$ must be trivial, by hypothesis, so $m^d \in kI$ for all $m \in M$. It follows that $x^d \in \mathbf{C}I$ for all $x \in M^{\#}$. But $M^{\#}$ is connected (by the remark before the Proposition) since M is also k-minimally almost periodic; thus $M^{\#}$ is a complex analytic abelian group, and its exponential mapping is surjective. Hence every element of $M^{\#}$ is a d-th power, so $M \subset M^{\#} \subset \mathbf{C}I$.

Now, if G is solvable, then $M = \pi(G)$ is solvable and quasibounded, so by (1.4) M contains an abelian subgroup A of finite index. Since M is also k-minimally almost periodic, $M^{\#}$ is connected by (2.3), so again by (2.3) A is Zariski dense in M. Thus M is abelian. On the other hand, if G is an almost connected k-Lie subgroup of $\mathrm{GL}(V)$, then we can identify the Lie algebra of M with a subalgebra of $\mathfrak{Gl}(W)$. Then Ad_M is a k-rational representation of M on a subspace of $\mathfrak{Gl}(W)$, hence $\mathrm{Ad}_M \circ \pi$ is a k-rational representation of G. But by (1.5.iii) applied to M, $\mathrm{Ad}_M(M) = (\mathrm{Ad}_M \circ \pi)(G)$ is compact and therefore trivial, by hypothesis. Thus M_0 is abelian. Again, by (2.3) and the hypothesis $G^{\#}$ is connected and G_0 is Zariski dense in G. Thus $\pi(G_0) \subset M_0$ is Zariski dense in $\pi(G) = M$, so M is abelian. ∎

Next, we turn to the (partial) converse of Theorem (2.4).

(2.6) THEOREM. *Let G be an almost connected k-Lie subgroup of* $\mathrm{GL}(V)$. *Suppose that for each closed subgroup H of G, if G/H carries a finite invariant measure then H is Zariski dense in G. Then G is k-minimally quasibounded.*

PROOF. By (2.5) it suffices to prove that if $\pi : G \to \mathrm{GL}(W)$ is a k-rational representation for which $\pi(G)$ is a bounded group, then $\pi(G) = \{I\}$. We deal separately with the real and complex cases. If $k = \mathbf{C}$, then π is an analytic homomorphism of the complex analytic group G_0 into a bounded group, hence $\pi(G_0)$ is trivial (see [9, p. 16] for example). Since G/G_0 is finite, the hypothesis implies that G_0 is Zariski dense in G, so $\pi(G)$ is also trivial.

To deal with the real case, we observe first that if S is a closed subgroup of

$\pi(G)$ for which the homogeneous space $\pi(G)/S$ is compact, then S is Zariski dense in $\pi(G)$. For by Lemma (1.8) both $\pi(G)$ and S are unimodular, so $\pi(G)/S$ and therefore also $G/\pi^{-1}(S)$ both carry finite invariant measures. Thus $\pi^{-1}(S)$ is Zariski dense in G by hypothesis, so S is Zariski dense in $\pi(G)$. Now by Proposition (1.5), $\pi(G)$ contains a vector subgroup A with $\pi(G)/A$ compact, so A is dense in $\pi(G)$. Thus it suffices to prove that $A = \{I\}$. Since A is a bounded vector group, it may be put simultaneously into the block diagonal form of (1.7), with each $\alpha = 0$ (since the eigenvalues must have modulus 1). If A is not trivial, then A acts like SO(2) on the two-dimensional subspace W_i corresponding to some non-trivial 2×2 block R_i; hence all orbits $A \cdot v$ in W_i are compact. If S_v is the stabilizer of v in A, then S_v is cocompact in A and in $\pi(G)$. It follows that S_v is Zariski dense in $\pi(G)$, so $\pi(G)$ fixes each W_i pointwise. This contradiction shows that no such 2×2 blocks exist, and that A is diagonalizable over $\mathbf{R}$. But A is connected and $U(n) \cap D(n, \mathbf{R})$ is discrete, so $A = \{1\}$.

(2.7) REMARKS.

(1) The result in (2.6) fails for discrete groups; in fact neither k-minimal quasiboundedness nor k-minimal almost periodicity is necessary for the density theorem when G is discrete. For example, if p is a prime and $G = \{R(n/p^k)$: $n, k \in \mathbf{Z}, k \geqq 1\}$ is the group of rotations in $\mathbf{R}^2$ by angles $2\pi n/p^k$, then G is an abelian group with no proper subgroups of finite index, so the density theorem holds vacuously; but G is a bounded group.

(2) There are analytic linear groups G for which some but not all of the closed proper subgroups satisfy the condition of the density theorem. For example, if G is the one-parameter group $(e^{tX})_{t \in \mathbf{R}}$ where

$$X = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \in \mathfrak{Gl}(2, \mathbf{R}) \qquad \text{with } \beta \neq 0,$$

then every closed subgroup H is of the form $(e^{n\theta X})_{n \in \mathbf{Z}}$ for some $\theta \in \mathbf{R}$. Thus G/H is compact for all H. On the other hand H is Zariski dense in G iff $\theta\beta/2\pi$ is irrational. Of course, this group G is not k-minimally quasibounded; quite the contrary, by (1.2) G is quasibounded (and not scalar).

The fact that the group G of (2) above is not k-minimally quasibounded goes hand in hand with the fact that the eigenvalues of elements of G are not real (even though, when $\alpha \neq 0$, the eigenvalues will lie off the unit circle, that is, G has a type E action). In fact, from (2.9) below it follows that among groups G for which $G^{\#}$ is connected (for example, when G is connected), G is k-minimally quasibounded iff G acts by real eigenvalues.

(2.8) LEMMA. *Let $k = \mathbf{R}$, and suppose $G \subset \mathrm{GL}(V)$ is triangularizable over $\mathbf{C}$. If each element of G has only real eigenvalues, then G is triangularizable over $\mathbf{R}$.*

PROOF. If w is a simultaneous eigenvector for G in $V_{\mathbf{C}}$, then $\bar{w}$ is also (where $\bar{w}$ is the complex conjugate of w relative to the canonical conjugation in $V_{\mathbf{C}}$). Thus $\mathrm{Re}\, w = (w + \bar{w})/2$, $\mathrm{Im}\, w = (w - \bar{w})/2i$ both generate G-stable subspaces in V, and since not both can be 0, G has a simultaneous eigenvector v in V. Now the proof can be completed by induction on $\dim V$, since the hypothesis is inherited by the image of G acting on $V/\mathbf{R}v$. ■

(2.9) PROPOSITION. *Let $k = \mathbf{R}$, and let $G \subset \mathrm{GL}(V)$ be solvable. Then G is k-minimally quasibounded iff $G^{\#}$ is connected and G acts by real eigenvalues on V.*

PROOF. ($\Leftarrow$): Suppose $\pi : G \to \mathrm{GL}(W)$ is an $\mathbf{R}$-rational representation with $M = \pi(G)$ bounded; by (2.5) it suffices to show that $M = \{I\}$. First, by (1.4) M contains an abelian subgroup of finite index; since $M^{\#} = \pi(G)^{\#} = \pi(G^{\#})$ is connected, it follows from (2.3) that M is abelian. Next we observe that M acts by real eigenvalues. For by the Lie–Kolchin theorem $G^{\#}$ is triangularizable over $\mathbf{C}$, so by (2.8) G is triangularizable over $\mathbf{R}$. Thus by continuity $G^{\#}$ is triangularizable over $\mathbf{R}$, that is, leaves stable an $\mathbf{R}$-rational flag in $V_{\mathbf{C}}$. By [2, (15.4) or (15.5)], $M^{\#} = \pi(G^{\#})$ is also triangularizable over $\mathbf{R}$, hence M acts by real eigenvalues. Since M is bounded and commutative, with real eigenvalues, it follows that M is in fact diagonalizable over $\mathbf{R}$; thus M is isomorphic to a bounded subgroup of $(\mathbf{R}^{\times})^{d}$, so M is finite. Now using (2.3) again we deduce that $M = \{I\}$.

($\Rightarrow$): We have already observed (2.2 and 2.3) that $G^{\#}$ is connected. Thus by the Lie–Kolchin theorem, $G^{\#}$ is triangularizable over $\mathbf{C}$, so the eigenvalues of $G^{\#}$ (and of G, of course) consist of the numbers $\chi(g)$, with χ a $\mathbf{C}$-rational character of $G^{\#}$, and $g \in G^{\#}$. If χ is such a character and $\varphi = \chi|_G$, then $\mathrm{Re}\,\varphi$ and $\mathrm{Im}\,\varphi$ are $\mathbf{R}$-rational functions on G, and

$$\pi : g \mapsto \begin{pmatrix} \mathrm{Re}\,\varphi(g) & -\mathrm{Im}\,\varphi(g) \\ \mathrm{Im}\,\varphi(g) & \mathrm{Re}\,\varphi(g) \end{pmatrix}$$

is an $\mathbf{R}$-rational representation of G into $\mathrm{GL}(2,\mathbf{R})$ with the $\pi(G)$ quasibounded: for π is the restriction of the (necessarily) homomorphic map on $G^{\#}$,

$$g \mapsto \begin{pmatrix} \sigma(g) & -\tau(g) \\ \tau(g) & \sigma(g) \end{pmatrix},$$

where σ and τ are $\mathbf{R}$-rational extensions to $G^{\#}$ of $\mathrm{Re}\,\varphi$, $\mathrm{Im}\,\varphi$, respectively. By hypothesis $\pi(G)$ consists of scalars, so $\chi|_G$ is real-valued. ■

(2.10) PROPOSITION. *Let G be a k-Lie subgroup of* $\mathrm{GL}(V)$. *Then G is k-minimally quasibounded in each of the following cases*:

(i) *G is minimally almost periodic*;

(ii) *$k = \mathbf{C}$, and G is complex analytic*;

(iii) *$k = \mathbf{R}$, G is real analytic, $R = \mathrm{rad}(G)$ acts by real eigenvalues, and G/R has no compact factors.*

PROOF. Let $\pi : G \to \mathrm{GL}(W)$ denote a k-rational representation of G such that $\pi(G)$ is quasibounded. If G is m.a.p., then $\det \circ \pi : G \to \mathbf{C}^{\times}$ must be trivial, so $\pi(G)$ is actually bounded ($\|\pi(x)\| = \omega(\pi(x))^{1/n}$); but then π itself must be trivial. This takes care of case (i). For cases (ii) and (iii) we may assume by (2.5) that $\pi(G)$ is bounded, and prove that π is trivial. Case (ii) has already been dealt with in the proof of (2.6). For case (iii), $R = \mathrm{rad}(G)$ is $\mathbf{R}$-minimally almost periodic by (2.9) and (2.5), so $\pi(R) = \{I\}$, and π induces a continuous homomorphism of the m.a.p. group $G/R \to \pi(G)$. By case (1) $\pi(G)$ is trivial. ∎

(2.11) REMARK. It may be worth pointing out that if $G = G_{\mathbf{R}}$ is an analytic group which is the group of real points of a solvable algebraic group, and if further no element of G has eigenvalues λ of modulus 1 except possibly $\lambda = 1$ itself (that is, if G has a type E action), then G is $\mathbf{R}$-minimally quasibounded. For in this case G cannot have any compact subgroups hence is simply connected, so G must act by real eigenvalues [9, (3.2)].

§3. In this section we give two applications of the Borel Density Theorem. Our first result concerns the Chabauty condition for lattices in a Lie group. If G is a separable, locally compact group, then the set of closed subgroups of G can be given a compact metrizable topology (the Chabauty topology) as follows: say that a sequence (H_n) of closed subgroups of G converges to the closed subgroup H if for each compact set $K \subset G$ and each neighborhood U of 1 in G, both the inclusions $H_n \cap K \subset HV$ and $H \cap K \subset H_n V$ hold for all sufficiently large n. This topology is described in more detail in [4], [3], [15]. In [4] Chabauty proved that when this topology is considered on the set of lattices in $\mathbf{R}^n$, it coincides on the $\mathrm{GL}(n, \mathbf{R})$-orbit of each lattice with the quotient topology from $\mathrm{GL}(n, \mathbf{R})$. In a series of papers [13], [14], [15], Wang has discussed other contexts to which one can (or cannot) generalize Chabauty's result. Here we prove an extension of Wang's result, proved in [13], that Chabauty's condition holds for simply connected nilpotent Lie groups.

(3.1) THEOREM. *Let G be a solvable analytic subgroup of* $\mathrm{GL}(n, \mathbf{R})$ *with only* *real eigenvalues, and let* Γ *be a lattice in* G. *Let* $\mathrm{Aut}(G)$ *denote the group of bicontinuous automorphisms of* G, *and let* $\mathcal{N}$ *denote the stabilizer of* Γ *in* $\mathrm{Aut}(G)$. *Then the Chabauty condition holds for* Γ: *the canonical bijection* $\mathrm{Aut}(G)/\mathcal{N} \to \mathrm{Aut}(G)\cdot\Gamma$ *is a homeomorphism (with the natural quotient topology on the former, and the Chabauty topology on the latter). In particular the orbits* $\mathrm{Aut}(G)\cdot\Gamma$ *are locally compact in the Chabauty topology.*

Proof. By results of Weil [16] (see [13, Theorem 9.4]) it suffices to prove that the cohomology restriction map $H^1(G, \mathfrak{G}) \to H^1(\Gamma, \mathfrak{G})$ is an isomorphism, where G acts on its Lie algebra $\mathfrak{G}$ by Ad. The proof now follows from Proposition (3.2) below. ∎

(3.2) Proposition. *Let* G *be a solvable analytic subgroup of* $\mathrm{GL}(n, \mathbf{R})$ *with only real eigenvalues, and let* H *be a closed uniform subgroup of* G *(equivalently, a closed subgroup for which* G/H *carries a finite invariant volume). Let* $\rho : G \to \mathrm{GL}(W)$ *be an* **R**-*rational representation. Then the cohomology restriction maps* $H^p(G, W) \to H^p(H, W)$ *are isomorphisms for all* $p \geqq 0$.

Proof. Since G is simply connected [9], the result will follow from Mostow's [12, Theorem 8.1] if we can show that H is ρ-ample in G, that is, that $(\rho \oplus \mathrm{Ad}_G)(H)$ is Zariski dense in $(\rho \oplus \mathrm{Ad}_G)(G)$. But H is Zariski dense in G by (2.9) and the density theorem (2.4). ∎

Our final result (3.5) is an application of the Borel density theorem to simple groups. It explains why in non-compact simple groups the only examples of subgroups with cofinite volume are lattices. It also generalizes the classical fact that the automorphism group of a compact Riemann surface of genus $g > 1$ is finite. The result will be deduced from (2.4), although it can in fact be deduced from Borel's original theorem [1].

(3.3) Proposition. *Let* G *be a* k-*minimally quasibounded Lie subgroup of* $\mathrm{GL}(V)$, *and let* H *be a closed subgroup of* G *such that* G/H *carries a finite invariant measure. Then any analytic subgroup* L *of* G *which is normalized by* H *is normal in* G.

Proof. L is normalized by H iff $\mathcal{L}$, the Lie algebra of L, is $\mathrm{Ad}_G(H)$-stable. Thus the lemma follows from (2.4).

As a generalization to non-linear groups, we have

(3.4) Proposition. *Let* G *be a* k-*Lie group, and* H *a closed subgroup of* G

such that G/H carries a finite invariant measure. If $\mathrm{Ad}_G(G)$ *is k-minimally quasibounded (in* $\mathrm{GL}(\mathfrak{G})$*) then any analytic subgroup L of G which is normalized by H is normal in G. In particular, this holds in all of the following cases:*

(i) *G is minimally almost periodic;*

(ii) *G is complex anlytic;*

(iii) *G is real analytic with radical R, G/R has no compact factors, and* $\mathrm{Ad}_G(R)$ *acts on* $\mathfrak{G}$ *with real eigenvalues.*

PROOF. L is normalized by H iff $\mathscr{L}$ is stable under $\mathrm{Ad}(H)$ iff $\mathscr{L}$ is stable under $\mathrm{Ad}(H)^-$, the Euclidean closure in $\mathrm{GL}(\mathfrak{G})$. Since $\mathrm{Ad}(G)/\mathrm{Ad}(H)^-$ has finite volume, the first statement follows from (2.4). The second statement now follows from (2.10). ■

(3.5) COROLLARY. *Let G be a non-compact simple analytic group, and let H be a closed subgroup of G such that $G \neq H$ and G/H carries a finite invariant measure. Then $N_G(H)$, the normalizer of H in G, is discrete. In particular, H is discrete and $N_G(H)/H$ is finite.*

PROOF. It will suffice to prove that $N = N_G(H)$ is discrete, since N/H has finite volume, and therefore is compact (as H is normal in N). Now N_0 is normal in N, and in particular is normalized by H, so by (3.4) N_0 is actually normal in G. Since G is simple, either $N_0 \subset Z(G)$, the (discrete) center of G, or $N_0 = G$. In the former case N_0 is discrete, hence so is N. The latter case, on the other hand, is impossible. For $N_0 = G$ implies that $N = G$; thus H is normal in G and G/H is a compact group. Moreover, H_0 is normal in G, so the hypothesis $H \neq G$ and simplicity imply that $H_0 \subset Z(G)$. But then H_0 and therefore H are discrete, so the compact group G/H is locally isomorphic to G, contradiction. ■

REFERENCES

1. A. Borel, *Density properties of certain subgroups of semisimple groups without compact components*, Ann. of Math. **72** (1960), 179–188.

2. A. Borel, *Linear Algebraic Groups*, Benjamin, New York, 1969.

3. N. Bourbaki, *Eléments de mathématique*, Livre VI, Chapitre 8, Hermann, Paris, 1963.

4. C. Chabauty, *Limite d'ensembles et géométrie des nombres*, Bull. Math. Soc. France **78** (1950), 143–151.

5. S. G. Dani, *A simple proof of Borel's density theorem*, Math. Z. **174** (1980), 81–94.

6. S. G. Dani, *On ergodic quasi-invariant measures of group automorphism*, Isr. J. of Math. **43** (1982), 62–74.

7. H. Furstenberg, *A note on Borel's density theorem*, Proc. Am. Math. Soc. **55** (1976), 209–212.

8. S. Helgason, *Differential Geometry and Symmetric Spaces*, Academic Press, New York, 1962.

9. M. Moskowitz, *On the density theorems of Borel and Furstenberg*, Ark. Math. **16** (1978), 11–27.

10. M. Moskowitz, *An abstract Borel density theorem*, Proc. Am. Math. Soc. **78** (1980), 19–22.

11. G. D. Mostow, *Fully reducible subgroups of algebraic groups*, Am. J. Math. **68** (1956), 200–221.

12. G. D. Mostow, *Cohomology of topological groups and solvable manifolds*, Ann. of Math. **73** (1961), 20–48.

13. S. P. Wang, *Limits of lattices in a Lie group*, Trans. Am. Math. Soc. **33** (1968), 519–526.

14. S. P. Wang, *On a conjecture of Chabauty*, Proc. Am. Math. Soc. **23** (1969), 569–572.

15. S. P. Wang, *On the limit of subgroups in a group*, Am. J. Math. **92** (1970), 708–724.

16. A. Weil, *Remarks on the cohomology of groups*, Ann. of Math. **80** (1964), 149–157.